Dynamic Estimation of the Hungarian Term Structure

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This paper focuses on dynamic properties of the Hungarian term structure. As Hungary is a key European emerging market empirical findings might offer value for both researchers and practitioners. The yield curve and its dynamics are first characterized by descriptive statistical analysis that is followed by a Principal Component Analysis (PCA). A semi non-parametric (SNP) study investigates structural dynamics of the yield curve without making parametric assumptions, then a stochastic mean reverting affine model (3-factor Vasicek model) is calibrated to the sample which is shown to work relatively more accurately in the Hungarian bond market than in the American one. The last section is devoted to forecasting future yield curves, where empirical results are somewhat less convincing.

Keywords: term structure of interest rates, affine model, Kalman filter, emerging markets

1. Introduction and motivation

Bonds represent claims for future cash flows, show the time value of money. The term structure of interest rates (yield curve) summarizes market expectations of a given time regarding the aforementioned time value of money; in our case shows how bond yields depend on time-to-maturity at each moment of time. It produces discount rates for risk-adjusted cash-flows in numerous financial applications.

Despite its key importance, the term structure is not directly observable. Therefore we need to estimate it. Term structure estimation evolved into two distinct though still related problems of finance. The first tries to produce a continuous yield curve on the back of some traded prices: this is the *static approach*. The curve is a snapshot of a given market, just as shown on Figure 1 depicting the Hungarian government bond market. The data source is the Hungarian Government Debt Management Agency (GDMA).

Static curve estimation is feasible via bootstrapping, Ordinary Least Squares (OLS) or Generalized Least Squares (GLS) and yield curve fitting techniques (e.g. cubic splines).

The second problem in finance, which this paper is devoted to, is related to the panel study approach and *focuses on dynamics of interest rates and the term structure*. The question is: how can we describe the evolution of interest rates over

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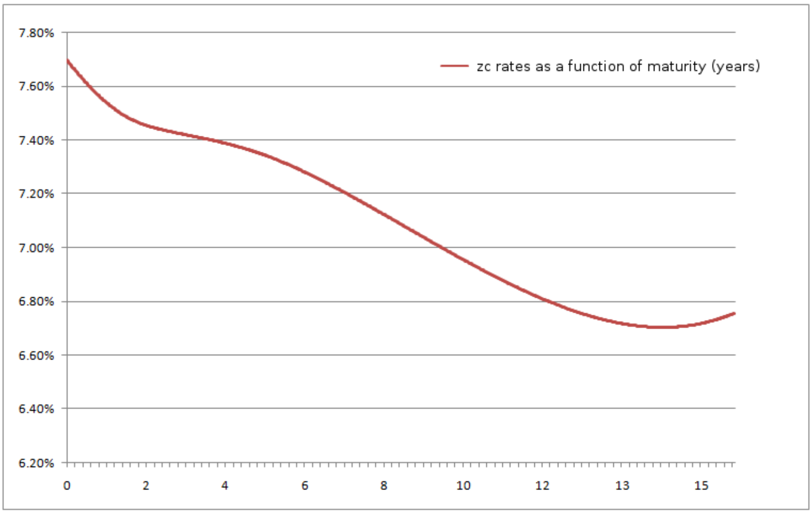
time? The concept is similar to how the evolution of a share price or a foreign exchange rate is estimated over time. Only similar, because the term structure – unlike share prices and FX rates – is not a scalar quantity. Different points (i.e. maturities) of the term structure cannot relate to each other in arbitrary ways, one must ensure that no-arbitrage rules apply.

Figure 1. GDMA zero coupon yield curve on 2 Jan 2008



Source: Government Debt Management Agency (GDMA)

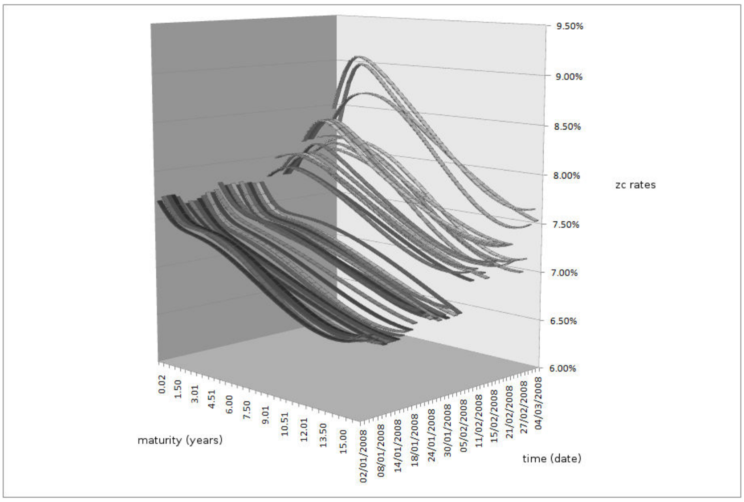
In the case of dynamic structured modelling estimation begins with selection of an interest rate model. To find an appropriate interest rate model is in itself a highly complicated issue, since there are dozens of frequently quoted models in literature. There is no universal interest rate model, therefore researchers often find their models as part of the estimation procedure (non-parametric estimation).

Estimation with structured models focuses on obtaining the distribution of the underlying stochastic variable(s) in the interest rate model. Shall this be infeasible (the pricing stochastic differential equation has no analytic solution) various moments of these distributions are estimated. The underlying stochastic variable is often not observable (e.g. volatility in models with several factors), therefore it has to be estimated as part of the estimation procedure. Only the sky and the lack of creativity limit the scope for empirical models.

The viewpoint of the dynamic approach is shown on Figure 2, displaying the evolution of the zero coupon yield curve as shown in Figure 1 over the period 2 January 2008 to 3 March 2008.

The *goal of this paper is to investigate Hungarian term structure dynamics via econometric methods*. My empirical models might help market actors *understand evolution of the yield curve* and *generate out-of-sample forecasts*. The *structure of this paper* is as follows. The first section presents descriptive statistical measures for the yield curve and a PCA model is run to guide stochastic modelling of the term structure, in terms of how many factors are truly needed for a proper fit. Then a semi non-parametric model is estimated to investigate yield curve dynamics without making any parametric assumptions. This is followed by stochastic modelling, where parametric assumptions are made and results of the PCA studies are facilitated. Here belongs the main conclusion of this paper: the 3-factor Vasicek model works particularly well for the Hungarian term structure. In the last section out-of-sample forecasting potential of the 3-factor Vasicek model is investigated.

Figure 2. GDMA zero coupon term structure between 2 Jan and 3 Mar 2008

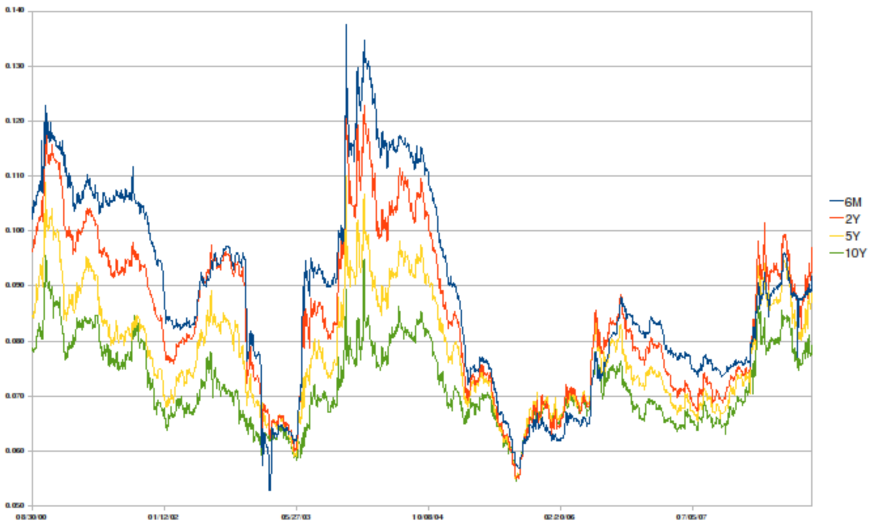


Source: Government Debt Management Agency (GDMA)

2. Descriptive statistics and PCA

For my empirical research I used a zero coupon sample of government bond data, collected on a daily basis between 1998 and 2008 by the GDMA. Recorded maturities are 2 week, 1 month, 3 month, 6 month, 9 month, 1 year, 2 year, 3 year, 4 year, 5 year, 6 year, 7 year, 8 year, 9 year and 10 year. Selected maturities are displayed on Figure 3.

Figure 3. Time series of sample data (N=2007)

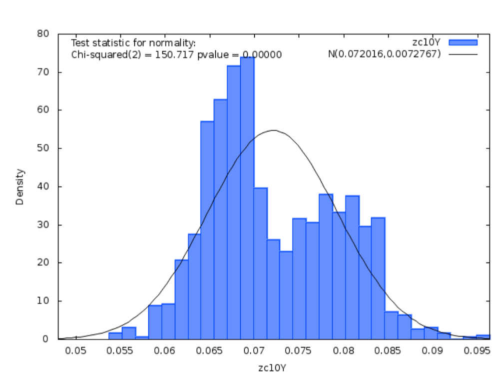


Source: Government Debt Management Agency (GDMA)

As the chart on Figure 3 shows, Hungarian interest rates have been volatile over the observation period. Also, the histogram of the 10 year tenor (shown on Figure 4) perfectly highlights the dual mode nature of the Hungarian market. It underscores a fact in a statistical way that is widely known by market participants from their own experience: things are either "very good" or "very bad" in the Hungarian bond market. "The pendulum often moves excessively into both extremes and only reverts with cumbersome slowness to standstill". The question whether that particular *standstill* can be regarded as fair value remains open to be answered by a macroeconomic study. If yes, we deal with excessive market reactions, if no, the bipolar nature of Hungarian economic processes warrant bimodal yield levels.

Figure 4. Descriptive statistics of the 10 year tenor

Average	0.07201
Median	0.06997
Minimum	0.05445
Maximum	0.09561
Standard deviation	0.00728
C. V.	0.10104
Skewness	0.35442
Ex. kurtosis	-0.64330



Source: own calculations

To judge how many factors are needed to describe Hungarian term structure dynamics, it is worth to carry out a Principal Component Analysis (PCA). After the famous paper of Litterman and Scheinkman (1991), it is common to assume that three factors, namely *level, steepness* and *curvature* drive the whole yield curve. Running a PCA on the Hungarian data set revealed that the first three factors cumulatively explain 99.81% of yield levels' covariance; results for daily yield changes show 95.71% cumulative explained covariance for the first three principal components.

Table 1. PCA of yield levels and their first differences

Principal components[2]	**Cumulated explained covariance (yield levels)**	**Cumulated explained covariance (daily yield changes)**
Factor 1	0.9470	0.6692
Factor 2	0.9933	0.8994
Factor 3	0.9981	0.9571
Factor 4	0.9994	0.9848
Factor 5	0.9998	0.9970

Source: own calculations

With the results shown in Table 1, we can conclude that with appropriate calibration, 3-factor stochastic models should produce a good fit for the Hungarian term structure.

2 Principal components are different in the two PCA studies.

3. Semi non-parametric model calibration

Over the course of semi non-parametric (SNP) analysis of the data sample, I started with plain VAR models (with lags 1, 2, 3 and 4), first these have been calibrated to the sample. Going further I continuously extended parameterisation of the auxiliary model (e.g. with ARCH and GARCH processes) and judged their significance by using two information criteria (AIC and BIC). Optimisations have been carried out with a method by Gallant and Tauchen (1996), using control runs to ensure robust results (i.e. not falling in the trap of a local minimum).

Results showed that Hungarian term structure dynamics are governed by a semi non-parametric GARCH process. The conditional variance of the auxiliary model is a VAR(1), GARCH(1,1) process, innovations are given by a 6th order polynomial with a time lag of 1. I investigated if using polynomials as coefficients of the innovation polynomial or introducing asymmetric volatility (leverage effect) into the model improve auxiliary model fit; but I did not get a confirmation in any of the two cases. SNP model calibration has been carried out for the tenors 6 month, 2 year, 5 year and 10 year in a pure time series approach, and for all these maturities combined in a panel approach. Different estimations confirmed each other.

My results for the Hungarian market are easily comparable with a study by Dai and Singleton (2000) referring to US markets. Authors there found the best score for a VAR(1), GARCH(1,2) auxiliary model, with innovations represented by a 4th order polynomial with a time lag of 1. All these let us conclude that *structural dynamics of the American and Hungarian yield curves are quite similar*, despite the fact that the Hungarian term structure had been an inverted one throughout the entire observation period. A partial explanation for this similarity might be, that *emerging market investors follow developed markets closely*, and *core market developments usually lead to important repercussions in emerging markets*.

4. Kalman filter calibration of the 3-factor Vasicek model

In this section, parametric assumptions are made, and a 3-factor stochastic model is calibrated to the sample. I chose the affine (i.e. constant plus linear) model family, and a mean reverting stochastic model by Vasicek (1977) for its simplicity and favourable applicability. In the Vasicek model, the vector of state variables, $X_{i,t}$ is driven by a mean reverting affine diffusion:

$$dX_{i,t} = \kappa_i (\theta_i - X_{i,t})dt + \sigma_i dW_{i,t},$$

for $i = 1, 2$ and 3. κ_i denote the strength of mean reversion, θ_i stand for long term factor means, σ_i mark factor volatilities and W_i are independent Wiener processes.

Model calibration has been carried out via the Kalman filter[3]: the likelihood function of the stochastic process has been reproduced with the Kalman filter, and this reproduced likelihood function has been estimated with maximum likelihood (ML).

In order to make an international comparison, I calibrated the 3-factor Vasicek model for the US bond market with a similar sample (daily observations of 15 maturities between 2001 and 2009), too. The results are shown in Table 2[4].

Table 2. Estimated parameters of the 3-factor Vasicek model

Parameter	Vasicek (HUF)	Vasicek (USD)
θ_1	0.000	0.020
θ_2	0.000	0.000
θ_3	0.000	0.039
κ_1	0.170	0.004
κ_2	0.675	0.246
κ_3	1.000	0.581
σ_1	0.022	0.007
σ_2	0.099	0.045
σ_3	0.033	0.012
λ_1	-0.217	-0.054
λ_2	-0.271	-0.383
λ_3	-0.330	-0.007
Average in-sample fit (bp)	8	5
Average sample yield level (%)	8.17	4.64
Relative model fit (bp per 100 bp yield level)	0.98	1.08

Source: own calculations

How do we interpret results? Comparing 3-factor model results in the Hungarian and American samples we find that: 1) long run factor means in the Hungarian sample are zero (θ_i parameters are bound to zero value due to admissibility issues), whereas in the US case we get numbers different from zero, these are the values where factors converge to. 2) The mean reversion process is stronger (κ_i values are higher) in the Hungarian sample than in the American case. 3) Estimated volatility parameters can be interpreted quite intuitively: factor volatilities in the emerging Hungarian

3 The Kalman filter, which is named after the Hungarian Rudolf Kalman, is particularly well suited to deal with problems where state variables (or a part of them) are latent. It involves an *a priori* estimate via system dynamics, then this prediction is combined with information gained from observations to refine the state estimate; the improved estimate is termed the *a posteriori* state estimate.

4 λ_i parameters characterize market prices of risk.

market are several-fold of those in the US treasury market which is the most important and also the most liquid in the world. 4) Average model mismatch (i.e. in-sample forecast residuals) seems to be slightly worse in the Hungarian sample, but if we normalize results with average yield levels, the *Hungarian model calibration turns out to be relatively more accurate than that for the American market*. Relative model fit (average in-sample residual per 1%p yield level) is 0.98 bp in the Hungarian market versus 1.08 bp in the American sample.

To demonstrate how well the 3-factor Vasicek model fits the sample, Figure 5 plots the 2 year maturity and its in-sample model estimate.

Figure 5. In-sample model fit of the 3-factor model (2 year tenor)

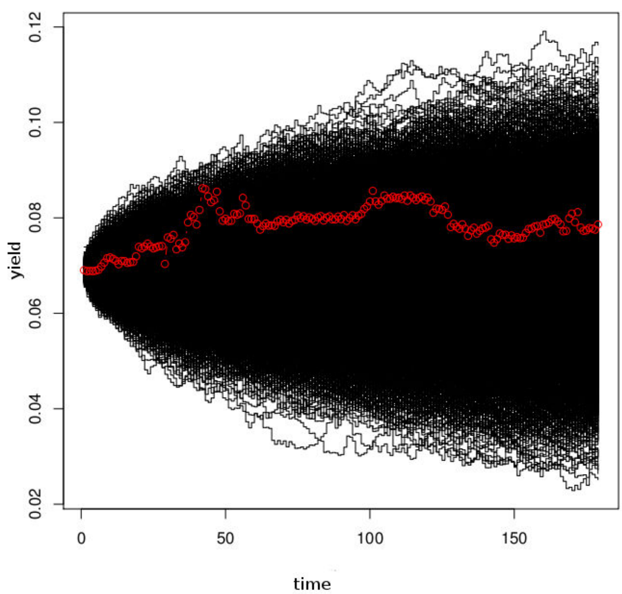
Source: own calculations

5. Forecasting future yield curves

With calibration of the 3-factor Vasicek model, I investigated its in-sample forecasting capabilities. Given the fact that the 3-factor model fit the Hungarian sample quite well, Hungarian term structure dynamics became easily understandable in a *retrospective way*. It remains an open question, though, whether the investigated model is capable to produce valuable out-of-sample forecasts. That constitutes a further step ahead, *"if we understand so much about the drivers of the yield curve, let us tell what the future holds"*.

Using a sample shortened by 180 days (i.e. $N' = 2007 - 180 = 1827$) I re-calibrated the 3-factor Vasicek model. The results, including average residuals, differed only marginally from original-sample-results, therefore I do not disclose them separately. Still, the use of a shortened sample makes the calibration process realistic and correct. With the resulting parameters, I ran simulations for the whole panel consisting of 15 maturities, producing N''=180 observations (i.e. the forecast horizon is 6 months). I "manufactured" missing term structure observations this way. I then repeated the simulation process 10 thousand times, to evaluate the model not on a single trajectory but rather on a fanchart-like, but empirically plotted graph. Comparing simulated trajectories with the last 180 original daily observations, we find how accurately the model would have forecasted future interest rates (*backtesting*). Average daily forecast errors have been saved for all trajectories, in order to generate an average measure for all 10 thousand trajectories. Figure 6 shows the probabilistic nature of realizations: simulated trajectories of the 10 year maturity are plotted with thin black lines (the darker trajectories are, the more likely they are to realize), last 180 original daily observations are dotted with red marks.

Figure 6. Simulated trajectories of the 3-factor Vasicek model (10 year tenor)



Source: own calculations

The average forecast error of the 10 year tenor on a 6 month forecast horizon is 108 bps (as shown in Table 3), however the mode of the distribution (i.e. the 10 thousand trajectories) is around 50 bps. That is an acceptable level for the volatile Hungarian market.

Still, Figure 6 underlines that out-of-sample forecasts have to be dealt with healthy cautiousness: the presented simulation is a valuable tool to forecast expected ranges of future interest rates, but it is definitely not an oracle to tell the winning market bet.

Table 3. Backtesting results based on 10 thousand trajectories

Maturity	**Average out-of-sample forecast errors (bp)**
2 week	282
1 month	279
3 month	270
6 month	258
9 month	247
1 year	241
2 year	215
3 year	190
4 year	173
5 year	159
6 year	144
7 year	134
8 year	123
9 year	116
10 year	108
Average of all tenors	196

Source: own calculations

Average backtesting errors are shown in Table 3, referring to 10 thousand simulated trajectories of all 15 maturities. It is clearly visible that *out-of-sample forecast errors are significantly higher* (25-fold more) *than their in-sample counterparts*. Also of note, is that *longer maturities* (i.e. less volatile parts of the yield curve) *are easier to predict*. In this example, I had the most accurate results with the 10 year tenor in the Hungarian sample. *Out-of-sample forecasting accuracy can be naturally improved by applying shorter forecast horizons*. With a forecast horizon of 1 week I got 13 bp average forecast error for the 10 year tenor, based on 10 thousand trajectories. This amounts approximately to 1.5-fold market bid-ask spread, i.e. it is a relatively acceptable result. Given that, the combined 2-hour runtime for the estimation and simulation algorithms is quite luring.

6. Concluding thoughts

This paper reaches the main conclusion that the 3-factor Vasicek model is a favourable choice for dynamic models on the Hungarian term structure. This statement is supported by empirical evidence regarding the model's in-sample forecasting potential. The 8 basis point average estimation error is first negligible with relation to the Hungarian market (practically it amounts to one unit bid-ask spread) and second reveals better relative (as corrected with average yield level) in-sample fit in the Hungarian market than in the US one.

When a researcher has to deal with structural modelling of the yield curve, they might consider the following points. It is sensible to choose an interest rate model for structural modelling which we have an efficient tool for to estimate. They shall avoid having a too complicated model which has to be calibrated by a non-linear estimation method which in turn cannot even recognize simple functional dependencies? On these grounds, I chose the affine model family and the Vasicek model. Decisions regarding the number of modelling factors are best guided by PCA studies. For empirical research on the Hungarian term structure I recommend the use of 3-factor models. Regarding estimation methods I had positive experience with the Kalman filter.

Considering out-of-sample forecasting potential of the 3-factor Vasicek model, one has to note that the model has somewhat limited potential for true forecasting purposes and results have to be interpreted by healthy cautiousness. The model is not an oracle to "tell the winning lottery draw", but a tool to show a range of expected future interest rates. As a practical note, less volatile maturities of the term structure should be used for forecasting purposes.

Concluding from the results detailed above the target audience of the presented methodology is rather the National Bank of Hungary, the Government Debt Management Agency and the Hungarian Financial Supervisory Authority. Commercial banks might find the methodologies useful, too; though their benefits are more likely to show up as more efficient risk management than hard profits of proprietary trading desks.

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