Dependence structure analysis using Copula-GARCH model: An application to UK and US stock markets

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The relationship between different international stock markets is of importance for both financial practitioners and academicians in order to manage risks. Especially after the financial crisis, the pronounced financial contagion draws the public attention to look into such associations. However, measuring and modelling dependence structure becomes complicated when asset returns present nonlinear, nongaussian and dynamic features. In this paper, we firstly investigate volatility spillover effect between FTSE100 and S&P500 stock indices. Strong lagged volatility of stock market itself and asymmetric spillover effect between UK and US stock markets are found out based on the multivariate GARCH-BEKK model. We also take a pilot study based on two step Copula-GARCH model to examine the correlation and tail dependence of returns. Some interesting results of co-movement between UK and US stock markets are discussed.

Keywords: Copula-GARCH, dependence structure, time-varying, volatility spillover effect

1. Introduction

Since the worldwide 2007- 2009 financial crisis or maybe even earlier, academicians and financial practitioners could not stop doubting the effectiveness of risk measurement and management of the financial market(s). It has been acknowledged that associations (co-movement) cross international stock market returns are of importance to measure and manage risks (Longin and Solnik, 1995). Dependence structure describes the relationship between risks and provides an estimation of risks. A deep understanding of dependence structure would help financial practitioners to

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make their financial decisions and financial regulators to control the financial contagion.

Traditional approaches measure dependence by linear correlation (namely Pearson's correlation coefficient) because linear correlation measure is straightforward to calculate. However, the use of linear correlation is only a symmetric, linear dependence metric (Embrechts *et al*. 2001). All the classical theories of risk measurement and management have been built on the primary assumption of multivariate normal independent identically distributed (i.i.d) return distributions. On contrary to the classical portfolio theory, modern asset returns present nonlinear, nongaussian and dynamic features. In addition, the ongoing unpredictable and changeable financial market scenario suggests that this kind of phenomena is likely to happen much more frequently. Accordingly, linear correlation measurement may induce misinterpretation when applied to nonlinear associations and tail dependency systems.

A powerful tool borrowed from mathematics namely copula functions (hereafter, referred as copulas) are capable to summarize the dependence structure between risks without the standard linear dependence and multivariate i.i.d return distribution restrictions. Copulas are defined as: functions that join one-dimensional distribution functions together to form multivariate distribution functions (Sklar 1959). Copulas not only solve the non-linear, non-elliptical problems, but also extend the bivariate world to multivariate dimension which gives us the ability to take all the risks into account as a whole unit. There were very few practical applications of copulas before the 1990s. As a consequence, the investigation of copulas has been restricted within the field of mathematics until the late 1990s when a number of statisticians showed an interest in implementation. Nelson (1999), who gave a thorough description of copulas from a mathematic perspective, is one notable example. Before too long, some academicians detected the potential of applying copulas in the field of finance. As soon as copulas and their applications were introduced; they became a favourite research trend of dependence in finance. Embrechts *et al* (1999, 2001) are among pioneers who modeled dependence with copulas and employed them to risk management. In terms of asset pricing with copulas, Cherubini *et al.* (2004) made a seminal contribution to the advent of pricing multivariate option by using copulas.

Jondeau, Rockinger (2006) advocated Copula-GARCH models which brought the advent of dynamic copula period. Copula-GARCH models are the class of models where some of copula parameters are potentially time-varying, in an autoregressive manner, conditional on the set of past information. Unlike the common normal and Archimedean copulas which have the constant parameters, Copula-GARCH models have time-varying dynamic parameters which would be naturally more suitable for financial time series data. Based on Copula-GARCH models, time varying conditional correlations will be provided rather than a constant correlation coefficiency over time period. In this case investors and policymakers are able to obtain the boundary of market fluctuation which would be helpful for them to analyse and diversify risk. Patton (2006a) proposed a practical two-stage maximum likelihood method to estimate copulas. Patton (2006b) introduced static and time-varying symmetric Joe-Clayton (SJC) copulas which allow different tail dependence. Aas *et al* (2009) introduce copula vines: a decomposition of a multivariate copula to a product of bivariate copulas.

In order to understand the linkages across the international stock markets, in this paper we tentatively investigate volatility spillover effect between FTSE100 and S&P500 stock indices. Strong lagged volatility of each stock market itself and asymmetric spillover effect cross different stock markets are found out based on the multivariate GARCH-BEKK model (Engle-Kroner, 1995). We employ a two step Copula-GARCH model to examine the dependence structure of daily stock markets returns. Firstly, we filter log-return daily data using univariate GARCH model to obtain standard residuals and construct the marginal distributions. Secondly, a couple of static and time-varying copulas are selected to join the estimated marginal distributions. The Akaike information criteria (AIC) and Bayesian information criteria (BIC) method are then adopted to determine which copula provides best fitness to the market data. Finally, some interesting results of co-movement between different stock markets are discussed.

2. Methodologies

2.1 Multivariate GARCH model (MV- GARCH)

It is a common belief that financial volatilities move together across markets over times. The GARCH model is regarded as an important implement to describe volatilities clustering of financial returns. MV-GARCH models extend the univariate GARCH model to multivariate dimensions which enable us to investigate spillover effects between different markets efficiently. Therefore we employ MV- GARCH model to analyse volatility movements between the UK and US stock markets. Engle and Kroner (1995) made improvement on the work of Baba, Engle, Kraft and Kroner (1988) and created a new simplified multivariate BEKK model which has been widely adopted quickly. The BEKK model can be represented as follows:

Mean equation

$R_t=C+\Theta R'_{t-1}+e_t \quad e_t \sim N(0, H_t)$ ……………………………………………..(1)

R_t is a T*1 vector, R_t is the daily logarithm return of each stock index. H_t is the variance-covariance matrix and it is constrained to be positive because the BEKK model uses a quadratic form for the parameter matrices to ensure a positive definite variance / covariance matrix H_t which is expressed as:

Variance-covariance

$$Ht=\Omega\Omega'+A\,(e_{t-1}e'_{t-1})A'+BH_{t-1}B' \qquad (2)$$

The error term e_t is defined as $e_t \sim N\,(0, H_t)$. It is suggested that student t distribution maybe the preferable one when we are dealing with the financial time series returns. Accordingly we redefine $e_t \sim ID\ (0, H_{t,}\ v)$ where v is degree of freedom and thus T-BEKK model is employed in this empirical study.

2.2 Copula-GARCH

2.2.1 Copulas

Copulas are defined as: functions that join one-dimensional distribution functions together to form multivariate distribution functions (Sklar 1959). Copulas not only solve the non-linear, non-elliptical problems, but also extend the bivariate world to multivariate dimension which gives us the ability to contain all the marginal distribution as a whole dependence structure. Generally speaking, there are two main types of copulas namely: implicit (elliptical) copula and Archimedean copulas. Each of them have their own family members and distinguishing properties. The elliptical copulas are known to perform better on systematic dependence structure problems. They constitute of Gaussian copula and student t copula. Archimedean copulas namely: Clayton copula and Gumbel copula captures lower tail dependence and upper tail dependence respectively (Alexander 2008). Empirical studies show that Archimedean copulas fit market data better than the elliptical copulas (Melchiori *et al*. 2003; Chen *et al. 2007,* Koziol-Kunisch 2005). We select Gaussian, Student t, Clayton and SJC copulas as candidates in the empirical study. The reasons are as following: firstly Gaussian copula is the standard copula tool although it does not consider the possibility of tail dependence which measures the joint probability of extreme events. Secondly, since the financial series data are known to have fat tails, Student t, Clayton and SJC copulas which all consider tail dependence are selected.

2.2.2 Estimation of Copula-GARCH

The method that we employ utilizes two stages maximum likelihood method to estimate Copula-GARCH. We start by constructing marginal distribution by fitting standard residuals data with unviariate GARCH model. Since financial time series return are known to be heteroscedastic and often autocorrelated, proper filtration is required. There are many types of conditional volatility GARCH models; in general they have been categorized as symmetric GARCH or asymmetric GARCH models. The latter satisfies daily data for equities(indices) and commodities at daily frequency because equity market volatility increases are more significant following a large negative return than when they are following a positive return of the same size which is so-called leverage effect(Alexander 2008). The GJR (Glosten, Jaganathan and Runkle, 1993) model modifies the conditional variance equation by introducing an extra factor—'leverage' parameter. Thus we use the GJR model which could reflect the asymmetric effect of how negative shock has greater impact on volatility than positive shock. The mathematical equations of the GJR are:

$$\varepsilon_t = h_t \bullet z_t, z_t \sim \mathrm{N}(0,1) \tag{3}$$

$$h_t = \varpi + \alpha \bullet \varepsilon_{t-1}^2 + \beta \bullet h_{t-1} + \gamma \bullet \varepsilon_{t-1}^2 \bullet \delta_{t-1} \tag{4}$$

where
$$\delta_t \text{ is equal to 1 when } \varepsilon_{\mathrm{t}} < 0$$
$$\delta_t \text{ is equal to 0 when } \varepsilon_{\mathrm{t}} > 0$$

The mean equation for AR (1)-GJR (1) is given by

$$r_t = c + s \bullet r_{t-1} + \varepsilon_t \tag{5}$$

Subsequently copulas are now used to construct the joint distributions. We follow Chen *et al* (2007) to select Kendall's *tau* and Spearman's *rho* as candidates of rank correlation statistics in our study. Then the correlation parameters corresponding to each copula are calculated based on the estimated Kendall's *tau*. Copulas estimation asks for uniform distribution data, the sample data should be transformed firstly. With the uniform distributed data, Maximum Likelihood Estimation (MLE) is then applied to estimate copula parameters:

$$L(\theta_c) = ArgMax_{\theta_c} \sum_{t=1}^{T} \ln C_t (F_1(x_{1t}), F_2(x_{2t}), ..., F_n(x_{nt}); \theta_c) \qquad (6)$$

where $F_n(x_{nt})$ are marginal distributions.

3. Data description

We choose the daily closed price from 5th Jan, 2004—31st, Sep, 2009 of FTSE100, and S&P500[3] stock indices as our observations. Logarithmic daily return of stock market (as a percentage) is defined as:

$$\gamma = 100 \ln(\frac{P_t}{P_{t-1}}) \qquad (7)$$

P_t is the daily closed price at time period t, basic statistics descriptive is shown in Table 1. According to the Jarque-Bera statistics, it is obvious that both of these two stock indices are non normal distributed. In particular, the FTSE100 is mildly right skewed while the S&P500 is negatively skewed. In addition, as evidenced by the large kurtosis value, the logarithmic daily return series show a strong leptokurtic feature. Furthermore, the results of Ljung-Box test reveal that returns series are also serially autocorrelated. The relationship between the mean and the median of FTSE100 provides possible evidence against the positive skewness. According to many standard text books the mean is usually located to the right of the median in the presence of positive skewness. However, in practise this rule often fails to apply. In particular for this FTSE100 return series distribution, areas to left and right of mean are not equal, and where one tail is long but when the other is heavy.

[3] Financial times stock index (FTSE100), Standard and Poor500 (SP500).

Table 1. Summary of Descriptive Statistics

	FTSE100	S&P500
Mean	0.008353	-0.006410
Median	0.054070	0.078815
Std. Dev.	1.395950	1.476607
Skewness	0.122954	-0.291525
Kurtosis	14.42809	13.23028
Jarque-Bera	7295.278	5862.420
Probability	0.000000	0.000000
Q(4)	35.37[0.39e-6]	34.08[0.72e-6]
Q(12)	89.56[0.00e-6]	54.34[0.24e-6]
Q^2 (4)	547 [0.00]	547.6 [0.00]
Q^2(12)	1176.3 [0.00]	1920.7 [0.00]

Source: own creation

4. Empirical results and analysis

4.1 The T-BEKK model

In the BEKK model, the conditional variance is not only a function of all lagged conditional variances and squared returns, but also a function of conditional covariance and cross-product returns. The diagonal elements in the parameter matrix B measures the effect of lagged volatility; the off-diagonal elements capture the cross market effects (Zhang 2009). We firstly carry out the bivariate T-BEKK estimation to examine the interactions between the UK and US stock markets.

Table 2. BEKK Estimation

FTSE100			S&P500		
	Estimates	**Standard Error**		**Estimates**	**Standard Error**
C_{11}	0.1929	0.21	C_{21}	-0.0576	0.026
C_{12}	0	0	C_{22}	0.0002	0.008
A_{11}	0.3192	0.0046	A_{21}	-0.4325	0.005
A_{12}	0.2281	0.0027	A_{22}	-0.0198	0.0031
B_{11}	0.841	0.0033	B_{21}	0.1119	0.002
B_{12}	-.0.0383	0.0041	B_{12}	0.9953	0.001
Degree of Freedom		10.589	**Log-likelihood**	-3510	

Source: own creation

First of all, the diagonal elements (B_{11}, B_{22}) all pass the 5% significance test which means strong lagged volatility exists in each stock market. Secondly, the volatility spillover effect is not symmetric. B_{12} (0.0383) and B21 (0.1119) suggest S&P500 transmit nearly 11.2% volatility shock to FTSE100 while only one third the other way around. It implies that S&P500 is dominating the volatility transmission effect between FTSE100 and S&P500.

4.2 Copula-GARCH

4.2.1 AR (1)-GJR model

The results of two marginal distributions are presented in Table 3. The value of the parameters for the variance in Eq. (4) and the return Eq. (5) display similar patterns. The leverage effect parameter γ are significant with estimates (0.16, 0.12) for FTSE100 and &P500 respectively. All the values are significantly positive, which indicates the existence of these effects. However, the leverage effects presenting in the U.S stock market seems to be weaker. It is worth to point out that persistence parameter β is significant for both indices which suggests that variances H_t rely on previous time period variance $H_{t-1.}$ Furthermore, the null value of ARCH (1) estimates can be explained by the fact that in the GJR model, the reaction effects have been taken into the leverage side.

Table3. Conditional marginal models estimation

	FTSE100		S&P500	
	Estimates	**Standard Error**	**Estimates**	**Standard Error**
C	0.021223	0.023123	0.0049888	0.026
AR(1)	-0.04504	0.0314	-0.07327	0.008
K	0.013301	0.0043661	0.012714	0.005
GARCH(1)	0.91904	0.015548	0.92962	0.0031
ARCH(1)	0	0.020629	0	0.0041
Leverage(1)	0.16191	0.032267	0.11397	0.001
Degree of Freedom	7.9116	2.0622	8.2209	1.4257

Notes: C is parameter c, AR (1) is the s, k is *w*, ARCH (1) is α, GARCH (1) is βand *r* is leverage effect respectively in Eq.(4) and Eq.(5) .
Source: own creation

4.2.2 Copula-GARCH estimation

With the conditional marginal model we are now in a position to estimate copulas. In this section we present the results of the six copulas estimation in Table 4. Compared with the copulas with constant parameters, the time-varying copulas have the lower negative log-likelihood which demonstrates that the time-varying copulas performance better. In other words, it tells us that the dependence structure is changing over the time since time-varying copulas improve the performance of the static copulas. Secondly, relied on the AIC, BIC value we conclude that the SJC copula is the most appropriate one.

Table 4. Log-likelihood Copula Estimation

FTSE100/SP500	Copula Type	Log-Likelihood	AIC	BIC
Constant	**Gaussian**	-216.346	-432.691	-432.687
	Student t	-218.65	-437.297	-437.289
	Clayton	-165.937	-331.873	-331.869
	SJC	-218.493	-436.981	-436.97
Time-varying	**Gaussian**	-218.443	-436.881	-436.87
	SJC	-221.998	-443.987	-443.964

Source: own creation

5. Conclusion and Future works

We examine the dependence structure between the UK and US stock market from two aspects: T-BEKK has been employed to investigate the volatility spillover effects between different markets. We find evidence that a volatility spillover effect significantly exists between the UK and US stock markets. Moreover the spillover effect is asymmetric; the S&P500 dominates the volatility effect. Furthermore Copula-GARCH methodology has been used to describe the dependence structure between different stock market. We conclude that the time-varying copulas perform better amongst all the six copula candidates. Especially the SJC time-varying copula accommodates difference in upper and lower tail dependences and so improves the log-likelihood of estimations. Future work will be done in two strands. Firstly, volatility spillover effects and dynamic conditional correlations among international stock markets are being investigated by a variety of multivariate GARCH models. Secondly, based on this pilot study work on GARCH-Copula applications to dynamic linkages between international financial markets is to be further developed. Moreover in light of the long, heavy tail feature of the return series tail dependences presented among different markets will be examined by SJC copula. Details and results would be seen in our forthcoming publications.

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